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A METHOD FOR SOLVING POLYNOMIAL EQUATIONS BY CONTINUED FRACTIONS

by

Amnon Bracha

July 1972

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A METHOD FOR SOLVING POLYNOMIAL EQUATIONS BY CONTINUED FRACTIONS

Amnon Bracha

Abstract

A method for the approximation of all the real roots of an n-order polynomial equation is developed. It is assumed that intervals containing the solutions are known. Bilinear transformations are used to approximate the solution. Convergence is achieved.

1. INTRODUCTION

In this paper we generalize earlier results by the same author [1] and develop them for finding all the zeroes of an n-order polynomial equation. In [1] it was shown that for a limited class of functions, such as quadratic or cubic equations, a solution can be approximated by a continued fraction of the form

$$\frac{A_k}{B_k} = \frac{p_1}{q_1} + \frac{p_2}{q_2} + \ldots + \frac{p_k}{q_k} \tag{1.1}$$

where A_k and B_k are determined from the recursion

$$\begin{aligned} A_i &= q_i A_{i-1} + p_i A_{i-2} \\ B_i &= q_i B_{i-1} + p_i B_{i-2} \end{aligned} \qquad i = 2, 3, \ldots \tag{1.2}$$

with initial values:

$$A_0 = 0 \qquad A_1 = p_1$$

$$B_0 = 1 \qquad B_1 = q_1 \ .$$

The digit set for p_i and q_i were selected as simple binary constants, e.g., 1/2 or 1, in order to reduce the amount of time required to evaluate (1.2).

In the current paper we show that polynomial equations of any order can be transformed by a bilinear transformation such as $x_i = \frac{p_i}{q_i + x_{i+1}}$ into another polynomial equation of the same order, where a simple recursion exists between the coefficients of the two polynomials.

The result is that if a method for selecting p_i and q_i for the i^{th} step can be developed, then we can approximate the solution by using recursion (1.2). The selection method is described in section 5.

In section 4 we develop a method for selecting two constants, a and b for p_i and q_i and we show the interval of solutions that can be approximated by these two constants. By using different pairs of constants we can, therefore, approximate different solutions of the given equation. Theorem 1 in the same section gives a proof of convergence, an important step in the development of the method. Rate of convergence is discussed in detail in [1].

2. BILINEAR TRANSFORMATIONS.

Following the analysis of Wynn [4] we define a continued fraction as a sequence of bilinear transformations of the form:

$$f_k = \frac{p_k}{q_k + f_{k+1}} , \qquad k = 1, 2, \ldots, \tag{2.1}$$

where $f_k(x)$ is a function of x and p_k, q_k are constants. The resulting continued fraction is

$$f_1 = \frac{p_1}{q_1} + \frac{p_2}{q_2} + \ldots + \frac{p_n}{q_n + f_{n+1}}$$

$$= \frac{A_n + f_{n+1} A_{n-1}}{B_n + f_{n+1} B_{n-1}} , \qquad n = 1, 2, \ldots, \tag{2.2}$$

where the functions A_i and B_i satisfy the recursion (1.2).

In a recent paper by the author [1] it was shown, following a note of Wynn [4], that the solution of the Riccati equation

$$y' + ay^2 + by + c = 0 , \tag{2.4}$$

where a, b, c are functions of x or constants, can be expanded as a continued fraction by using a series of bilinear transformations of the form (2.1), and such that each function f_k, $k = 1, 2, \ldots$ also satisfies the Riccati equation. In [1] the recursions for the k^{th} Riccati equation were developed and a method for selecting p_k, q_k for each step was shown for several functions.

In the current paper we develop the results of [1] for polynomials of any order, and we show how the method can be used to find the zeroes of these polynomials.

One important assumption in the development of the method in [1] was that p_k, q_k, are simple binary constants, i.e. 1 or 1/2, and therefore the various recursions that are involved require only "short" operations. In the current paper we generalize this result and include other values for p_k and q_k.

3. THE SOLUTION OF A POLYNOMIAL AS A CONTINUED FRACTION.

We show now how distinct roots of a given polynomial can be approximated by using bilinear transformations.

First we develop the recursion for the coefficients of the polynomial for the $i+1^{th}$ step by using the coefficients of the i^{th} step, $i = 1, 2, \ldots$.

Let

$$P_n^1(x) = \sum_{k=0}^{n} a_k^1 x_1^k = 0 \tag{3.1}$$

be a given polynomial equation of order n. The index 1 refers to the first step in the recursion. In particular, the index 1 is <u>not</u> an exponent, whereas the index k is an exponent when used as a superscript.

We use a substitution of the form

$$x_i = \frac{p_i}{q_i + x_{i+1}}, \qquad i = 1, 2, \ldots, \tag{3.2}$$

where p_i, q_i are constants to be defined.

Suppose that the i^{th} step polynomial equation is of the form

$$P_n^i = a_n^i x_i^n + a_{n-1}^i x_i^{n-1} + \ldots + a_1^i x_i + a_0^i = 0 , \tag{3.3}$$

then by using substitution (3.2) we have

$$a_n^i \left(\frac{p_i}{q_i+x_{i+1}}\right)^n + a_{n-1}^i \left(\frac{p_i}{q_i+x_{i+1}}\right)^{n-1} + \ldots + a_1^i \frac{p_i}{q_i+x_{i+1}} + a_0^i = 0 .$$

Multiply by $(q_i + x_{i+1})^n$ to normalize the coefficient of a_n^i. We get

$$a_n^i p_i^n + a_{n-1}^i p_i^{n-1}(q_i+x_{i+1}) + \ldots + a_1^i p_i (q_i+x_{i+1})^{n-1} + a_0^i (q_i+x_{i+1})^n = 0 .$$

The recursion that follows is:

$$
\begin{aligned}
a_n^{i+1} &= a_0^i \\
a_{n-1}^{i+1} &= a_1^i p_i + n a_0^i q_i \\
&\cdot \\
&\cdot \\
&\cdot \\
a_k^{i+1} &= \sum_{t=0}^{n-k} \binom{k+t}{t} a_{n-k-t}^i p_i^{n-k-t} q_i^t \qquad k = 0, 1, \ldots, n.
\end{aligned}
\tag{3.4}
$$

The resulting $i+1^{th}$ step polynomial equation is

$$P_n^{i+1} = a_n^{i+1} x_{i+1}^n + a_{n-1}^{i+1} x_{i+1}^{n-1} + \ldots + a_1^{i+1} x_{i+1} + a_0^{i+1} = 0 .$$

The method to approximate the solution of (3.1) can be used now, if an algorithm for selecting p_i and q_i ($i = 1, 2, \ldots$) is defined. This is the subject of section 5.

4. INTERVALUES THAT ARE COVERED BY CONTINUED FRACTIONS

In this section we show how to approximate values of the solution by using different pairs of constants p_i and q_i.

Let p_i and q_i each assume two given values, e.g., $p_i, q_i \in \{a,b\}$. Continued fractions of the form (1.1) assume all values in the interval $[M,m]$, where

$$M = \max \lim_{k\to\infty} \frac{A_k}{B_k} \quad ,$$

$$m = \min \lim_{k\to\infty} \frac{A_k}{B_k} \quad .$$

A simple analysis shows that M can be found by solving a quadratic equation of the form

$$M = \cfrac{p_{mx}}{q_{min} + \cfrac{p_{min}}{q_{mx} + M}}$$

where p_{mx}, $q_{mx} = \max(a,b)$, and p_{min}, $q_{min} = \min(a,b)$.

Similarly we have for m

$$m = \cfrac{p_{min}}{q_{mx} + \cfrac{p_{mx}}{q_{min} + m}} \quad .$$

The resulting equations are:

$$q_{min} M^2 + (q_{min}\, q_{mx} + p_{min} - p_{mx})\, M - p_{mx}\, q_{mx} = 0$$

and

$$q_{mx} m^2 + (q_{mx}\, q_{min} + p_{mx} - p_{min})\, m - p_{min}\, q_{min} = 0 \, .$$

Assuming that both p and q have the same range, e.g., $p_{min} = q_{min} = a$, $p_{mx} = q_{mx} = b$, $p,q \in \{a,b\}$ then the equation has the form

$$aM^2 + (ab + a - b)\, M - b^2 = 0$$
$$bm^2 + (ab + b - a)\, m - a^2 = 0 \quad . \tag{4.1}$$

In the following table we give values of a, b and the corresponding ranges [m,M].

a	b	m	M
1/2	1	$\frac{\sqrt{2}-1}{2}$	$\sqrt{2}$
1	2	$\frac{\sqrt{17}-3}{4}$	$\frac{\sqrt{17}-1}{2}$
-4	-6	1	2

If a solution of the given polynomial equation is known to be in a certain interval, then the appropriate digit set a, b can be used to approximate this solution.

5. SELECTION RULES

In this section we develop a method for selecting p_i and q_i for the case $0 < a < b$, m and M positive. With some modifications the analysis can be developed for other cases. Assume that a pair of digits, $0 < a < b$, were selected according to the analysis of section 4. The range of the solution, e.g., x_1, can be found now by solving (4.1). Since we are using a substitution of the form (3.2) our first condition will be:

$$m \leq x_i \leq M \qquad i = 1, 2, \ldots \tag{5.1}$$

By imposing condition (5.1) we need now only one set of selection rules for p_i and q_i, $i = 1, 2, \ldots$, in the range $[m,M]$.

We write below a version of (3.1). Let

$$x_1 = \frac{-a_0^1}{\sum_{k=0}^{n} a_k^1 x_1^{k-1}} \,. \tag{5.2}$$

where it is assumed that $m \leq x_1 \leq M$.

We will find p_1 and q_1 such that

$$x_1 = \frac{p_1}{q_1 + x_2} \tag{5.3}$$

where $m \leq x_2 \leq M$ and $p_1, q_1 \in \{a,b\}$. Clearly we have four possibilities for selecting p_1 and q_1 and for each such pair we get different x_2. In order to make our selection we adopt the inverse approach. We assume that condition (5.1) exists for x_2, and find the range for x_1 for each pair of p_1 and q_1.

We start with the pair $p_1 = a$, $q_1 = b$. From (5.3) we have

$$C = \frac{a}{b+m} \geq x_1 \geq \frac{a}{b+M} = c \tag{5.4}$$

Since a, b, m and M are known, C and c can be found and we have defined a range for x_1 for which a selection of $p_1 = a$ and $q_1 = b$ will assure the condition (5.1). Since x_1 is an unknown we substitute the results of (5.4) in (5.2) in order to find the allowable range for $p_1 = a$ and $q_1 = b$. We have

$$c \leqq \frac{-a_0^1}{\sum_{k=0}^{n} a_k^1 x_1^{k-1}} \leqq C$$

and this result is possible for any x_1 in the range [c,C], therefore we conclude that if the following two conditions are satisfied

$$\sum_{k=0}^{n} a_k^1 C^k \geqq 0$$

(5.5)

$$\sum_{k=0}^{n} a_k^1 c^k \leqq 0 \ ,$$

we select $p_1 = a$ and $q_1 = b$.

The analysis can be carried now for each of the remaining three pairs of values of p and q. Since x_2 has the same range as x_1 and satisfies a polynomial of the same degree, we can use the same procedure for x_2, etc.

The result is that the entire range is divided into four sections. For each section we can choose a pair of p_i and q_i, such that condition (5.1) will be satisfied for x_{i+1}.

Clearly, by using only the upper bound in (5.5), for each pair of p_i and q_i we can define a unique set of selection rules.

Our next objective is to show that there is an overlapping between two consecutive regions so that by using only the upper bound for each pair, the entire region is covered.

It can be verified that the four regions defined by (5.5) for each pair p_i and q_i are:

$$
\begin{aligned}
&(a) \quad p = b, \qquad q = a \\
&(b) \quad p = b, \qquad q = b \\
&(c) \quad p = a, \qquad q = a \\
&(d) \quad p = a, \qquad q = b .
\end{aligned}
\tag{5.6}
$$

In the theorem which follows we give a necessary and sufficient condition, for the overlapping of the regions defined in (5.6), for two cases. The analysis for other values of a and b is similar and therefore is omitted.

Theorem 1:

The regions defined by (5.5), for each of the pairs (5.6) overlap each other if and only if the following conditions are satisfied:

(a) If $0 < a < b$, then $M \cdot m \leq 1/2$.

(b) If $b < a < 0$, then $M \cdot m \geq 1/2$.

Proof: For $0 < a < b$ the regions defined in (5.6) are in decreasing order. Therefore we only have to show that the upper bound for the pair (5.6)-(b),(c),(d) is greater or equal to the lower bound of (5.6)-(a),(b),(c) respectively.

For the upper bound of (5.6)-(b) and lower bound of (5.6)-(a), we have,

$$\frac{b}{b+m} \geq \frac{b}{a+M} , \tag{5.7}$$

for the second pair we have,

$$\frac{a}{a+m} \geq \frac{b}{b+M}$$

and for the third

$$\frac{a}{b+m} \geqq \frac{a}{a+M} \quad . \tag{5.8}$$

The conditions that must be satisfied are:

$$\begin{aligned} M - m &\geqq b - a \\ M \cdot a &\geqq m \cdot b \quad . \end{aligned} \tag{5.9}$$

From the definition of a,b, the range [m,M] and condition (5.1) we have:

$$\frac{b}{a+m} = M \quad \text{and} \quad \frac{a}{b+M} = m \quad .$$

Eliminating a and b we get

$$a = \frac{mM(1+m)}{1-mM}$$

and

$$b = \frac{mM(1+M)}{1-mM}$$

The first condition in (5.9) is satisfied if and only if

$$M - m \geqq b - a = \frac{mM(1+M-m-1)}{1-mM} = \frac{mM(M-m)}{1-mM}$$

or

$$mM \leqq 1/2 \quad .$$

For the second condition we have,

$$\frac{M^2m(1+m)}{1-mM} \geqq \frac{m^2M(1+M)}{1-mM}$$

or

$$M \geqq m$$

which was assumed.

For the second case, $b < a < 0$, the inequalities in (5.7) and (5.8) reverse and therefore the result follows.

6. CONVERGENCE CONDITIONS

In this section we develop necessary and sufficient conditions for the procedure to converge.

Theorem 2:

Let

$$x_i = \frac{A}{B} = \frac{p_1}{q_1} + \frac{p_2}{q_2} + \ldots \quad \frac{p_n}{q_n + x_{n+1}} \tag{6.1}$$

be a continued fraction representation of a solution of (3.1) which was found by substitution (3.2). $p_i, q_i \in \{a,b\}$, $x_i \in [m,M]$, $(i=1,2,\ldots)$, where relations (4.1) exist between a, b, m and M.

Necessary and sufficient conditions for (6.1) to converge to a solution of the polynomial equation (3.1) are:

(a) $\frac{q}{p}(2m+a) > 0$

(b) $2 + \frac{q}{p}(a+m-M) > 0$

Proof: Let A_n/B_n be the n^{th} approximation to x_1. We will show that under condition (a) and (b), for every $\epsilon > 0$, there exists N, such that for all $n > N$

$$\delta_n = \frac{A}{B} - \frac{A_n}{B_n} < \epsilon \ ,$$

or equivalently that if $T_n = \delta_n/\delta_{n-2}$ $n = 3,4,\ldots$ then under the conditions of the theorem $|T_n| < 1$.

It was already shown in [1] that

$$T_n = \frac{1 - \frac{q_n}{p_n} x_n}{1 + \frac{q_n}{p_n} \frac{B_{n-1}}{B_{n-2}}} \ ,$$

therefore the condition $|T_n| < 1$ implies:

$$1 - \frac{q_n}{p_n} x_n < 1 + \frac{q_n}{p_n} \frac{B_{n-1}}{B_{n-2}}$$

and

$$-1 - \frac{q_n}{p_n} \frac{B_{n-1}}{B_{n-2}} < 1 - \frac{q_n}{p_n} x_n \quad .$$

For the first expression we have:

$$\frac{q_n}{p_n} \left(x_n + \frac{B_{n-1}}{B_{n-2}}\right) > 0 \ ,$$

and since this condition is true for every x_n it can be replaced by m, also $\min \frac{B_{n-1}}{B_{n-2}} = a+m$ and the result is:

$$\frac{q}{p} (a+2m) > 0 \ .$$

For the second expression we have

$$2 + \frac{q_n}{p_n} \left(\frac{B_{n-1}}{B_{n-2}} - x_n\right) > 0$$

here we substitute M for x_n and the minimum value of $\frac{B_{n-1}}{B_{n-2}}$, and the second condition follows:

$$2 + \frac{q}{p} (a+m-M) > 0 \quad .$$

7. CONCLUSIONS

A method for the approximation of real roots of an n-order polynomial equation is described. The assumption is that intervals, each containing one solution are known. Those intervals are later used according to the analysis of section 4 as m and M in order to define a pair of digits a and b, which will be used in a bilinear transformation such as (3.2) to approximate the solution.

The algorithm described in this paper consists of the steps:

(1) Select a digit set, a and b, which covers one solution at a time.

(2) Use iteration (2.3) to approximate the solution.

(3) Iterate on (3.4) to get the coefficients of the $i+1^{th}$ polynomial from the i^{th} polynomial equation.

(4) Use selection rules, such as (5.5), which covers the entire allowable region, for the next p_i and q_i.

(5) Check for accuracy and if obtained divide A_k by B_k to receive one solution.

(6) Repeat (1) to (5) for each interval which contains a zero of the given polynomial equation.

Notes:

(a) If the condition of theorem 1 is not satisfied for a certain interval, this interval can be divided into subintervals.

(b) By using powers of 2 for p_i and q_i, the time required to evaluate most of the expressions is reduced if a binary computer is used.

(c) An example of a polynomial equation of order 5 is given in appendix A.

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APPENDIX A

We show now how to approximate one solution of a given polynomial equation of order 5:

Let

$$P_5^1(x) = (x+3)(x+2)(x-1)(x-2)(x-3)$$

$$= x^5 - x^4 - 13x^3 + 13x^2 + 36x - 36 = 0$$

be a given polynomial equation of order 5, and suppose that it is known that the interval

$$\frac{\sqrt{17}-3}{4} , \frac{\sqrt{17}-1}{2}$$

contains one solution.

Our first step is to select a pair of digits, a and b, according to the analysis of section 4, such that every value in the given interval can be approximated by a continued fraction, $p,q \in \{a,b\}$. From the table at the end of the section we have: $a = 1$, $b = 2$.

The recursion relation between the coefficients of the i^{th} and $i+1^{th}$ polynomials of order 5 are:

$$a_5^{i+1} = a_0^i$$

$$a_4^{i+1} = a_1^i + 5a_0^i q_i$$

$$a_3^{i+1} = a_2^i p_i^2 + 4a_1^i p_i q_i + 10a_0^i q_i^2$$

$$a_2^{i+1} = a_3^i p_i^3 + 3a_2^i p_i^2 q_i + 6a_1^i p_i q_i^2 + 10a_0^i q_i^3$$

$$a_1^{i+1} = a_4^i p_i^4 + 2a_3^i p^3 q_i + 3a_2^i p_i^2 q_i^2 + 4a_1^i p_i q_i^3 + 5a_0^i q_i^4$$

$$a_0^{i+1} = a_5^i p_i^5 + a_4^i p_i^4 q_i + a_3^i p_i^3 q_i^2 + a_2^i p_i^2 q_i^3 + a_1^i p_i q_i^4 + a_0^i q_i^5$$

The selection rules according to the analysis of section 5, are:

$$\text{for } P_5^i\left(\frac{5-\sqrt{17}}{2}\right) \geqq 0, \qquad p_i = 1, \quad q_i = 2;$$

$$\text{for } P_5^i\left(\frac{\sqrt{17}-1}{4}\right) \geqq 0, \qquad p_i = 1, \quad q_i = 1;$$

$$\text{for } P_5^i\,(5-\sqrt{17}) \geqq 0, \qquad p_i = 2, \quad q_i = 2;$$

$$\text{Otherwise} \quad p_i = 2, \quad q_i = 1\ .$$

k	p_k	q_k	A_k	B_k	A_k/B_k	Error
1	2.C	1.C	C.C	C.1CCCD C1	C.C	C.10000D 01
2	2.0	1.0	C.200CD C1	0.1000D 01	C.2CCCCCCCCCCCCCCCD C1	-C.1C00CD C1
3	2.0	1.C	C.2CCCD C1	0.3CCCD C1	C.6666666666666666D 0C	C.33333D CC
4	2.0	1.0	C.60CCD C1	C.5CCCD C1	C.12CCCCCCCCCCCCCCCD 01	-C.2CCCCD 00
5	2.0	1.C	C.1CCCD C2	0.1100D 02	C.9C9C9C9C9C9O9O9O91D CC	C.9C9C9D-C1
6	2.C	1.C	C.22CCD C2	C.21CCD C2	C.1C476190476190476D 01	-C.47619D-C1
7	2.0	1.C	C.42CCD C2	C.43CCD C2	C.9767441E6C465116D CC	C.23256D-C1
8	2.C	1.C	C.86CCD C2	0.85CCD 02	C.1C117647C58823553D C1	-C.11765D-C1
9	2.0	1.C	C.17CCD C3	C.171CD C3	C.994152C467E36257D CC	C.58480D-02
1C	2.0	1.C	0.342CD C3	C.341CD C3	C.1CC293255131964ED C1	-C.29326D-C2
11	2.C	1.C	C.682CD C3	C.683CD 03	C.998535871156661BD CC	C.14641D-C2
12	2.C	1.C	C.1366D C4	0.1365D C4	C.1CCC7326CC7326C1D C1	-0.7326CD-03
13	2.C	1.C	C.2730D 04	0.2731D C4	C.9996338337EC5273D CC	C.36617D-C3
14	2.C	1.C	C.5462D C4	C.5461D C4	C.1CCC183116645303D C1	-0.18312D-C3
15	2.0	1.C	C.1C92D C5	C.1C92D C5	C.9999C8450C555C75D CC	C.9155CD-C4
16	2.0	1.C	C.2185D C5	0.2185D C5	C.1CCCC45777C656905D C1	-C.45777D-C4
17	2.0	1.C	C.4369D C5	C.4369D C5	C.9999771119910279D CC	C.22888D-04
18	2.0	1.C	C.8738D C5	C.8738D C5	C.1CCCO11444135453D C1	-C.11444D-C4
19	2.C	1.C	C.1748D C6	C.1748D 06	O.9999942779650155D CC	C.5722CD-C5
20	2.C	1.C	C.3495D C6	0.3495D C6	C.1CCCC02861025678D C1	-0.2861CD-C5
21	2.0	1.C	C.6991D C6	0.6991D C6	C.9999985659E4E92C75D CC	0.143C5D-C5
22	2.C	1.C	C.1398D C7	0.1398D C7	C.1CCOOOO715255590BD 01	-C.71526D-06
23	2.C	1.C	C.2796D C7	0.2796D C7	C.9999996423721740D CC	0.35763D-06
24	2.C	1.C	C.5592D C7	0.5592D C7	C.1CCCOCC178813545D C1	-C.17881D-C6
25	2.C	1.C	C.1118D C8	0.1118D C8	C.9999995105930355D 0C	C.89407D-C7
26	2.0	1.0	0.2237D C8	C.2237D C8	C.1CCCCCCC447C3484D C1	-C.44703D-07
27	2.0	1.C	0.4474D C8	0.4474D 08	C.9999999776482583D CC	C.22352D-C7
28	2.0	1.C	C.8948D C8	0.8946D C8	C.1CCCOOOO111758710 01	-C.11176D-C7
29	2.0	1.C	0.1790D C9	C.1790D C9	C.9999999954412C645D CC	C.55879D-C8
3C	2.C	1.C	C.3579D C9	0.3579D C9	C.1CCCCOOOC2753968D C1	-C.2794CD-08
31	2.C	1.C	C.7158D C9	C.7158D C9	C.9999999986030161D 0C	C.13970D-08
32	2.0	1.0	0.1432D 10	0.1432D 1C	C.1CCCCCOCCC6584920 C1	-C.6584CD-09
33	2.C	1.C	C.2863D 10	0.2863D 10	0.9999999996SC754CD CC	0.34925D-C9
34	2.C	1.C	C.5727D 1C	0.5727D 1C	C.1CCCCCCCCC0174623D C1	-0.17462D-09
35	2.C	1.C	0.1145D 11	0.1145D 11	C.9999999995512E885D CC	C.87311D-1C
36	2.C	1.C	C.2291D 11	C.2291D 11	C.1CCCCCOCCOC43656D C1	-C.43656D-1C
37	2.0	1.0	C.4581D 11	0.4581D 11	C.9999999997817210 0C	C.21828D-1C
38	2.0	1.C	C.9163D 11	0.9163D 11	C.1CCCCCOOOOC1C514D C1	-C.1C514D-1C
39	2.C	1.C	C.1833D 12	C.1833D 12	C.9999999999945430D 0C	0.5457CD-11
40	2.0	1.C	C.3665D 12	C.3665D 12	C.1CCCCCCCCCCC2728D C1	-C.27285D-11
41	2.C	1.C	0.7330D 12	0.7330D 12	0.9999999999986357D CC	C.13642D-11
42	2.C	1.C	C.1466D 13	C.1466D 13	C.1CCCCCCCCCCCCC682D C1	-C.68212D-12
43	2.0	1.C	0.2932D 13	0.2932D 13	C.9999999999995658SD CC	0.34106D-12
44	2.C	1.C	C.5864D 13	0.5864D 13	C.1CCCCCOOOOOOO170D C1	-C.17053D-12
45	2.C	1.C	C.1173D 14	C.1173D 14	C.9999999999999147D CC	0.85265D-13
46	2.C	1.C	0.2346D 14	0.2346D 14	C.1CCCCCCCCCCCOO43D C1	-C.42633D-13
47	2.C	1.C	C.4691D 14	C.4691D 14	C.9999999959999787D 0C	0.21316D-13
48	2.0	1.C	C.9382D 14	C.9382D 14	C.1CCCCCCCCCCCCC1CD C1	-0.1C658D-13
49	2.C	1.C	0.1876D 15	0.1876D 15	C.9999999999995947D 0C	0.53291D-14
50	2.0	1.C	C.3753D 15	C.3753D 15	C.1CCCCOOOCCCOOOC3D C1	-C.26645D-14

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